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Entangled Quantum States

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Entangled quantum states are an important component of quantum computing techniques such as quantum error correction, dense coding, and quantum teleportation. We determine the requirements for a state in the Hilbert space $\mathbf{C}^9$ to be entangled and a solution to the corresponding factorization problem if this is not the case.

Entanglement is the characteristic trait of quantum mechanics which entails its departure from classical physics. We consider the entanglement of pure states. Thus a basic question in quantum computing is as follows: given a normalized state $|\mathbf{u}\rangle$ in the Hilbert space $\mathbf{C}^9$, can two normalized states $|\mathbf{x}\rangle$ and $|\mathbf{y}\rangle$ in the Hilbert space $\mathbf{C}^3$ be found such that

$$|\mathbf{x}\rangle \otimes |\mathbf{y}\rangle = |\mathbf{u}\rangle \tag{1}$$

where $\otimes$ denotes the Kronecker product [1, 2]. In other words, what is the condition on $|\mathbf{u}\rangle$ such that $|\mathbf{x}\rangle$ and $|\mathbf{y}\rangle$ exist? If no such $|\mathbf{x}\rangle$ and $|\mathbf{y}\rangle$, exist then $|\mathbf{u}\rangle$ is said to be *entangled*. The measure for entanglement for pure states $E(\mathbf{u})$ is defined as follows [3, 4]:

$$E(\mathbf{u}) := S(\rho_A) = S(\rho_B)$$

where the density matrices are defined as

$$\rho_A := \mathrm{Tr}_B|\mathbf{u}\rangle\langle\mathbf{u}|, \qquad \rho_B := \mathrm{Tr}_A|\mathbf{u}\rangle\langle\mathbf{u}|$$

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and

$$S(\rho) := -\mathrm{Tr}\,\rho \log_2 \rho$$

Thus $0 \le E \le 1$. If $E = 1$, we call the pure state maximally entangled. If $E = 0$, the pure state is not entangled. For example, the vector

$$\frac{1}{3}(1\ 1\ 1\ -1\ -1\ -1\ 1\ 1\ 1)^T = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

is not entangled, whereas the vector

$$\frac{1}{\sqrt{2}}(1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1)^T$$

is entangled.

First we we derive the condition for a normalized state in $\mathbf{C}^9$ to be not entangled, using the Kronecker product in Eq. (1). Then we derive for this case that $E(\mathbf{u}) = 0$ follows. We use the representation

$$|\mathbf{u}\rangle = \begin{pmatrix}u_1\\u_2\\u_3\\u_4\\u_5\\u_6\\u_7\\u_8\\u_9\end{pmatrix}, \qquad |\mathbf{x}\rangle = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, \qquad |\mathbf{y}\rangle = \begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}$$

where $|\mathbf{u}\rangle$, $|\mathbf{x}\rangle$, and $|\mathbf{y}\rangle$ are normalized, i.e., $\langle\mathbf{u}|\mathbf{u}\rangle = \langle\mathbf{x}|\mathbf{x}\rangle = \langle\mathbf{y}|\mathbf{y}\rangle = 1$. Since $|\mathbf{u}\rangle$ is normalized, at least one of $u_1, u_2, \ldots, u_9$ is nonzero. The same holds for $|\mathbf{x}\rangle$ and $|\mathbf{y}\rangle$. From the normalization conditions and (1) we find

$$\sum_{k=1}^{9} |u_k|^2 = 1 \tag{2}$$

$$|x_1|^2 + |x_2|^2 + |x_3|^2 = 1 \tag{3}$$

$$|y_1|^2 + |y_2|^2 + |y_3|^2 = 1 \tag{4}$$

$$x_1 y_1 = u_1, \qquad x_2 y_1 = u_4, \qquad x_3 y_1 = u_7 \tag{5}$$

$$x_1 y_2 = u_2, \qquad x_2 y_2 = u_5, \qquad x_3 y_2 = u_8 \tag{6}$$

$$x_1 y_3 = u_3, \qquad x_2 y_3 = u_6, \qquad x_3 y_3 = u_9 \tag{7}$$

From (5)–(7) we find that the condition on $|\mathbf{u}\rangle$ is given by

$$u_1 u_5 = u_2 u_4 \tag{8}$$

$$u_1 u_6 = u_3 u_4 \tag{9}$$

$$u_2 u_6 = u_3 u_5 \tag{10}$$

$$u_4 u_8 = u_5 u_7 \tag{11}$$

$$u_4 u_9 = u_6 u_7 \tag{12}$$

$$u_5 u_9 = u_6 u_8 \tag{13}$$

$$u_1 u_8 = u_2 u_7 \tag{14}$$

$$u_1 u_9 = u_3 u_7 \tag{15}$$

$$u_2 u_9 = u_3 u_8 \tag{16}$$

From (8)–(16) we find

$$u_1 u_5 u_9 = u_1 u_6 u_8 = u_2 u_4 u_9 = u_2 u_6 u_7 = u_3 u_4 u_8 = u_3 u_5 u_7 \tag{17}$$

This equation is useful for simplification. From (3)–(7) we obtain

$$|x_1|^2 = |u_1|^2 + |u_2|^2 + |u_3|^2 \tag{18}$$

$$|x_2|^2 = |u_4|^2 + |u_5|^2 + |u_6|^2 \tag{19}$$

$$|x_3|^2 = |u_7|^2 + |u_8|^2 + |u_9|^2 \tag{20}$$

$$|y_1|^2 = |u_1|^2 + |u_4|^2 + |u_7|^2 \tag{21}$$

$$|y_2|^2 = |u_2|^2 + |u_5|^2 + |u_8|^2 \tag{22}$$

$$|y_3|^2 = |u_3|^2 + |u_6|^2 + |u_9|^2 \tag{23}$$

Let

$$\alpha_1 := \arg(x_1), \qquad \alpha_2 := \arg(x_2), \qquad \alpha_3 := \arg(x_3)$$

$$\beta_1 := \arg(y_1), \qquad \beta_2 := \arg(y_2), \qquad \beta_3 := \arg(y_3) \tag{24}$$

Now Eq. (5)–(7) become

$$\alpha_1 + \beta_1 = \arg(u_1) \mod 2\pi \tag{25}$$

$$\alpha_1 + \beta_2 = \arg(u_2) \mod 2\pi \tag{26}$$

$$\alpha_1 + \beta_3 = \arg(u_3) \mod 2\pi \tag{27}$$

$$\alpha_2 + \beta_1 = \arg(u_4) \mod 2\pi \tag{28}$$

$$\alpha_2 + \beta_2 = \arg(u_5) \mod 2\pi \tag{29}$$

$$\alpha_2 + \beta_3 = \arg(u_6) \mod 2\pi \tag{30}$$

$$\alpha_3 + \beta_1 = \arg(u_7) \mod 2\pi \tag{31}$$

$$\alpha_3 + \beta_2 = \arg(u_8) \mod 2\pi \tag{32}$$

$$\alpha_3 + \beta_3 = \arg(u_9) \mod 2\pi \tag{33}$$

Suppose that (8)–(16) hold; then a solution is given by

$$x_1 = (\sqrt{|u_1|^2 + |u_2|^2 + |u_3|^2})\, e^{i\alpha_1} \tag{34}$$

$$x_2 = (\sqrt{|u_4|^2 + |u_5|^2 + |u_6|^2})\, e^{i\alpha_2} \tag{35}$$

$$x_3 = (\sqrt{|u_7|^2 + |u_8|^2 + |u_9|^2})\, e^{i\alpha_3} \tag{36}$$

$$y_1 = (\sqrt{|u_1|^2 + |u_4|^2 + |u_7|^2})\, e^{i\beta_1} \tag{37}$$

$$y_2 = (\sqrt{|u_2|^2 + |u_5|^2 + |u_8|^2})\, e^{i\beta_2} \tag{38}$$

$$y_3 = (\sqrt{|u_3|^2 + |u_6|^2 + |u_9|^2})\, e^{i\beta_3} \tag{39}$$

$$\begin{array}{lll} \alpha_1 = 0, & \alpha_2 = \arg(u_4) - \beta_1, & \alpha_3 = \arg(u_7) - \beta_1 \\ \beta_1 = \arg(u_1), & \beta_2 = \arg(u_2), & \beta_3 = \arg(u_3) \end{array} \tag{40}$$

Next we describe the relation between conditions (8)–(16) and the measure of entanglement $E(\mathbf{u})$ introduced above. Since

$$|\mathbf{u}\rangle\langle\mathbf{u}| = \begin{pmatrix} u_1\overline{u_1} & u_1\overline{u_2} & u_1\overline{u_3} & u_1\overline{u_4} & u_1\overline{u_5} & u_1\overline{u_6} & u_1\overline{u_7} & u_1\overline{u_8} & u_1\overline{u_9} \\ u_2\overline{u_1} & u_2\overline{u_2} & u_2\overline{u_3} & u_2\overline{u_4} & u_2\overline{u_5} & u_2\overline{u_6} & u_2\overline{u_7} & u_2\overline{u_8} & u_2\overline{u_9} \\ u_3\overline{u_1} & u_3\overline{u_2} & u_3\overline{u_3} & u_3\overline{u_4} & u_3\overline{u_5} & u_3\overline{u_6} & u_3\overline{u_7} & u_3\overline{u_8} & u_3\overline{u_9} \\ u_4\overline{u_1} & u_4\overline{u_2} & u_4\overline{u_3} & u_4\overline{u_4} & u_4\overline{u_5} & u_4\overline{u_6} & u_4\overline{u_7} & u_4\overline{u_8} & u_4\overline{u_9} \\ u_5\overline{u_1} & u_5\overline{u_2} & u_5\overline{u_3} & u_5\overline{u_4} & u_5\overline{u_5} & u_5\overline{u_6} & u_5\overline{u_7} & u_5\overline{u_8} & u_5\overline{u_9} \\ u_6\overline{u_1} & u_6\overline{u_2} & u_6\overline{u_3} & u_6\overline{u_4} & u_6\overline{u_5} & u_6\overline{u_6} & u_6\overline{u_7} & u_6\overline{u_8} & u_6\overline{u_9} \\ u_7\overline{u_1} & u_7\overline{u_2} & u_7\overline{u_3} & u_7\overline{u_4} & u_7\overline{u_5} & u_7\overline{u_6} & u_7\overline{u_7} & u_7\overline{u_8} & u_7\overline{u_9} \\ u_8\overline{u_1} & u_8\overline{u_2} & u_8\overline{u_3} & u_8\overline{u_4} & u_8\overline{u_5} & u_8\overline{u_6} & u_8\overline{u_7} & u_8\overline{u_8} & u_8\overline{u_9} \\ u_9\overline{u_1} & u_9\overline{u_2} & u_9\overline{u_3} & u_9\overline{u_4} & u_9\overline{u_5} & u_9\overline{u_6} & u_9\overline{u_7} & u_9\overline{u_8} & u_9\overline{u_9} \end{pmatrix} \tag{41}$$

we find

$$\rho_A := \mathrm{Tr}_B(|\mathbf{u}\rangle\langle\mathbf{u}|)$$

$$= \begin{pmatrix} u_1\overline{u_1} + u_2\overline{u_2} + u_3\overline{u_3} & u_1\overline{u_4} + u_2\overline{u_5} + u_3\overline{u_6} & u_1\overline{u_7} + u_2\overline{u_8} + u_3\overline{u_9} \\ u_4\overline{u_1} + u_5\overline{u_2} + u_6\overline{u_3} & u_4\overline{u_4} + u_5\overline{u_5} + u_6\overline{u_6} & u_4\overline{u_7} + u_5\overline{u_8} + u_6\overline{u_9} \\ u_7\overline{u_1} + u_8\overline{u_2} + u_9\overline{u_3} & u_7\overline{u_4} + u_8\overline{u_5} + u_9\overline{u_6} & u_7\overline{u_7} + u_8\overline{u_8} + u_9\overline{u_9} \end{pmatrix} \tag{42}$$

$$\rho_B := \mathrm{Tr}_A\,(|\mathbf{u}\rangle\langle\mathbf{u}|)$$

$$= \begin{pmatrix} u_1\overline{u_1} + u_4\overline{u_4} + u_7\overline{u_7} & u_1\overline{u_2} + u_4\overline{u_5} + u_7\overline{u_8} & u_1\overline{u_3} + u_4\overline{u_6} + u_7\overline{u_9} \\ u_2\overline{u_1} + u_5\overline{u_4} + u_8\overline{u_7} & u_2\overline{u_2} + u_5\overline{u_5} + u_8\overline{u_8} & u_2\overline{u_3} + u_5\overline{u_6} + u_8\overline{u_9} \\ u_3\overline{u_1} + u_6\overline{u_4} + u_9\overline{u_7} & u_3\overline{u_2} + u_6\overline{u_5} + u_9\overline{u_8} & u_3\overline{u_3} + u_6\overline{u_6} + u_9\overline{u_9} \end{pmatrix} \quad (43)$$

The 3×3 density matrices ρ_A and ρ_B given by (42) and (43) are Hermitian and have the same eigenvalues. Thus the eigenvalues λ_1, λ_2, and λ_3 are real. The matrices are also positive semidefinite, i.e., for all $|\mathbf{a}\rangle \in \mathbf{C}^3$ we have $\langle\mathbf{a}|\rho_{A,B}|\mathbf{a}\rangle \geq 0$. Thus the eigenvalues are nonnegative. Since $|\mathbf{u}\rangle$ is normalized we have

$$\mathrm{Tr}(\mathrm{Tr}_A|\mathbf{u}\rangle\langle\mathbf{u}|) = 1 \quad (44)$$

$$\mathrm{Tr}(\mathrm{Tr}_B|\mathbf{u}\rangle\langle\mathbf{u}|) = 1 \quad (45)$$

and therefore

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 \quad (46)$$

where we used the fact that the trace of an $n \times n$ matrix is the sum of the eigenvalues. Thus $0 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1$. Now if Eq. (17) holds, we have

$$\det(\mathrm{Tr}_A(|\mathbf{u}\rangle\langle\mathbf{u}|)) = \det(\mathrm{Tr}_B(|\mathbf{u}\rangle\langle\mathbf{u}|)) = 0 \quad (47)$$

Since the determinant of an $n \times n$ matrix is the product of the eigenvalues we find that at least one eigenvalue is equal to 0. Since, using (8)–(16),

$$\begin{aligned} \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 &= (\rho_A)_{1,1}(\rho_A)_{2,2} + (\rho_A)_{1,1}(\rho_A)_{3,3} + (\rho_A)_{2,2}(\rho_A)_{3,3} \\ &\quad -(\rho_A)_{1,2}(\rho_A)_{2,1} - (\rho_A)_{1,3}(\rho_A)_{3,1} - (\rho_A)_{2,3}(\rho_A)_{3,2} \\ &= (\rho_B)_{1,1}(\rho_B)_{2,2} + (\rho_B)_{1,1}(\rho_B)_{3,3} + (\rho_B)_{2,2}(\rho_B)_{3,3} \\ &\quad - (\rho_B)_{1,2}(\rho_B)_{2,1} - (\rho_B)_{1,3}(\rho_B)_{3,1} - (\rho_B)_{2,3}(\rho_B)_{3,2} \\ &= 0 \end{aligned} \quad (48)$$

two eigenvalues must be zero. We applied SymbolicC++ [5] to do this calculation. The last eigenvalue is 1. The entanglement can be written as

$$E(\mathbf{u}) = -(\lambda_1 \log_2 \lambda_1 + \lambda_2 \log_2 \lambda_2 + \lambda_3 \log_2 \lambda_3) \quad (49)$$

Using $\log_2 1 = 0$, $0 \log_2 0 = 0$, we find that $E(\mathbf{u}) = 0$ if conditions (8)–(16) are satisfied. Conversely, we can prove that if $E(\mathbf{u}) = 0$, conditions (8)–(16) follow. The proof can be extended from 3 to n dimensions.

For $\lambda_1 = \lambda_2 = \frac{1}{2}$, $\lambda_3 = 0$ (and permutations) the entanglement $E(\mathbf{u})$ has a maximum and we find $E(\mathbf{u}) = 1$. As an example, consider the normalized state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1)^T$$

Obviously Eq. (15) does not hold for $|\psi\rangle$. We find

$$\rho_A = \rho_B = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho_A \log_2 \rho_A = -\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $S(\rho_A) = 1$. Thus this state is maximally entangled.

Finally, we mention that Horodecki *et al.* [6] and Peres [7] investigated whether a given mixed state is entangled (inseparable) or nonentangled (separable), using the density matrix ρ. They started from the following definition: a state supported on a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is separable if and only if it can be written in (or approximated by) the form

$$\rho = \sum_{i=1}^{k} p_i |e_i\rangle \otimes |f_i\rangle\langle e_i| \otimes \langle f_i|, \qquad \sum_{i=1}^{k} p_i = 1$$

where $|e_i\rangle$ and $|f_i\rangle$ are normalized states in the Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, and $p_i \geq 0$ $(i = 1, \ldots, n)$.

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